DOCUMENT RESUME

ED 173 106 SE 027 913

AUTHOR Schlissel, Arthur
TITLE Slowly and Rapidly Converging Infinite Series.
PUB DATE [79]
NOTE 5p.

EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS *Calculus; *College Mathematics; Concept Formation; Higher Education; *Learning Activities; *Mathematics Instruction; Teaching Techniques
IDENTIFIERS *Estimation (Mathematics); *Sequences (Mathematics)

ABSTRACT

A method is presented for constructing infinite series with prescribed rates of convergence. The method is presented as being an aid to improving the student's understanding of how the partial sum of an infinite series can be used to approximate the sum of the series, as well as being of interest in itself. The focus of the paper is the degree of approximation obtained by using a partial sum for the sum of the infinite series, since this concept is of great importance in using a partial sum for approximation purposes and is often left vague in conventional approaches. (MP)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

ED173106

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY Arthur Schlissel TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

Slowly and Rapidly Converging Infinite Series

Arthur Schlissel

John Jay College of Criminal Justice - C.U.N.Y.

In this paper a method is presented for constructing infinite series with prescribed rates of convergence which is of some interest in itself and will improve the student's understanding of how the partial sum of an infinite series can be used to approximate the sum of the series. An important part of any calculus course is the concept of infinite series, their convergence, divergence and their use in approximations. But the usual concern when teaching infinite series is with their convergence and divergence. To this end the Cauchy convergence criterion is introduced followed by the ratio and root tests. These are then applied to various series to determine their convergence or divergence. The student is thus left with the notion that the partial sums of a convergent series approach its sum. But the degree of approximation obtained by using a partial sum for the sum of the infinite series itself is often left vague. This concept is however of great importance in using a partial sum for approximation purposes. This paper is an attempt to clarify this concept for the student by presenting a method for constructing an infinite series having an arbitrarily prescribed rate of convergence, either slow or rapid.

U.S. DEPARTMENT OF HEALTH, EDUCATION & WELFARE NATIONAL INSTITUTE OF EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

SE 027 913

If the series $\sum_{j=1}^{\infty} a_j$ is convergent with sum S and partial sums $S_n = \sum_{j=1}^{n} a_j$ it follows from the definition of convergent series that for any $\varepsilon > 0$ there can be found an integer N such that $|S - S_n| < \varepsilon$ for all $n \geq N$. In other words, the partial sums S_n can be made arbitrarily close to the sum S by taking a "sufficient number" of terms of the series. We next introduce the "rate of convergence" of an infinite series as the function f(n): $f(n) = |S - S_n|$. We now show how to construct the terms of a series $\sum_{j=1}^{\infty} a_j$ with the sum S, such that the partial sums $S_n = \sum_{j=1}^{n} a_j$ converge at a prescribed rate to S as $n \to \infty$. Let f(x) be a monotonically decreasing function for $0 \leq x < \infty$ with f(0) = S, which converges to zero at a specified rate as $x \to \infty$. From the definition of f(n) we obtain $S - S_n = f(n)$ and $S - S_{n+1} = f(n+1)$. Subtracting the first expression from the last one and using the definition of S_n yields the $n^{\underline{th}}$ terms of the infinite series: $a_n = f(n+1)$, $(n=1,2,\ldots)$. Hence the partial sums $S_n = \sum_{j=1}^{n} a_j$ converge to S as $n \to \infty$ at the same rate with which f(x) converges to 0 as $x \to \infty$. The resulting series has the required property i.e. the partial sums S_n converge to the limit S slowly or rapidly as a function of n as n increases. [1]

[1] Infinite series with a prescribed sum, but without any regard to rate of convergence, are constructed in [2, p. 233].

Functions which yield slowly or rapidly converging infinite series can easily be constructed by use of the scale of infinite described in [1, Ch. 1]. The functions $f(x) = \frac{S \log(3)}{\log(x+3)}$ and $f(x) = \frac{S \log\log k}{\log\log(x+k)}$, $k = 3^3$, will yield slowly converging series. The form of the numerator is dictated by the requirement that $f(0) = S$ and that the numerator be positive. (Instead of 3 any number greater than the natural logarithm base e could also be used.) By iterating the logarithm in the above examples a more slowly converging series is produced. [2] Examples of functions which yield rapidly converging infinite series are $f(x) = Se^{-x}$ and $f(x) = Se^{-x^2}$. Here the presence of S is dictated by the requirement that $f(0) = S$. By increasing the exponent of x in the above expression one obtains a more rapidly convergent series. For possibilities of constructing other "slowly" and more "rapidly" decreasing functions f(x) see [1, Chap. 3]. The above examples yield series where all terms a_n are positive. These functions can also be used to construct alternating series, e.g.,

$$g(x) = \frac{S \log(3)}{\log(x+3)} \sin\left(\frac{\pi}{2} + \pi x\right).$$

The number of terms to obtain a desired degree of approximation for the series constructed above are easily found as follows. For any small ε, $\varepsilon > 0$, set $f(x) = \varepsilon$ and choose n as the largest integer contained in x, then $|S - S_n| < \varepsilon$.

[2] Theorems which show that for any convergent infinite series another infinite series which converges more slowly can be constructed are found in [2, p. 301]. But these results are theoretical and not constructive in nature.

Thus n gives the number of terms necessary for the required degree of approximation.

Bibliography

[1] Hardy, G. H., "Orders of Infinity," Cambridge Pub. Co., 1954.

[2] Knopp, K., "The Theory of Infinite Series," Hafler Pub. Co. (2 Eng. Ed.), 1971.